Int. J. Solids Structures, 1977, Vol. 13, pp. 159–178. Pergamon Press. Printed in Great Britain

THE PATCH TEST AND THE GENERAL CONVERGENCE CRITERIA OF THE FINITE ELEMENT METHOD

E. R. DE ARANTES E OLIVEIRA
Departamento de Engenharia Civil, Universidade Tecnica de Lisboa, Avenida Rovisco Pais, Lisboa 1, Portugal

(*Received* 10 *March* 1975; *revised* 6 *May* 1976)

Abstract—A synthetic description of continuous structural models followed by some general ideas on hybrid discrete models is first presented. An approximation theorem is then established which leads to a general expression of an upper bound of the discretization error, equally dependent on the strain and displacement interpolation errors, as well as on the exact and approximate solutions. The last part of the paper is devoted to the justification of the patch test with the help of such expression. It becomes clear that passing the patch test is not a necessary condition for convergence and that the simple patch test is sufficient for accuracy and not merely for convergence analysis.

1. INTRODUCTION

Several authors have been writing on convergence and accuracy in the finite element method for the last ten years, and the author himself wrote a number of paper concerning structural and non-structural cases and the role of convergence in the mathematical theory of structures.

The problem of convergence is not really a difficult one if the elements are such that conformity is achieved because, then, completeness implies convergence. The completeness criterion being not sufficient, however, in the non-conforming case, finding supplementary criteria for such general situation has not proved to be an easy matter.

Some authors, like recently Myoshi[1], Nitsche[2] and Kikushi[3], do not seem to have been especially interested in general criteria. Establishing general expressions for the discretization error and applying it to particular elements was indeed their main concern.

General criteria are useful however, not because they suppress the particularization to a given kind of element, which has always to be done as the final step of any convergence analysis, but because they make it simpler.

The patch test was one of the most significant advances in convergence analysis just because it provided a general criterion for the non-conforming case. Passing the patch test together with completeness is indeed sufficient for convergence.

The test was devised by Irons[4] as an empirical tool and first studied by Strang[5] from a mathematical point of view. Other mathematitians like Ciarlet[6] have used it in their work on convergence.

The author's own ideas on convergence and accuracy are expounded in the present paper and the patch test justified with the help of such ideas.

The first part of the paper is devoted to a synthetic description of continuous structural models, followed by some general ideas on hybrid discrete models generated by the potential energy method. The corresponding class of elements, which comprehends Pian's hybrid elements[7] as a particular case, represents the most general case of kinematically non-conforming elements; those within which displacements and strains are not connected by the continuous homogeneous strain-displacement equations. A dual class of models characterized by statical non-conformity can of course be generated by the complementary energy method but is not considered in the present paper.

An approximation theorem is then stated which leads to a general expression of the error or, more precisely, of an upper bound of the error (see inequality (5.16)), which is quite different from the ones given by the authors quoted above.

An important point about about such expression is its symmetry: the error becomes indeed equally dependent on the strain and on the displacement interpolation errors, on the exact and on the approximate solutions. It becomes clear moreover that the order of magnitude of the error depends on the boundedness of certain derivatives of both solutions, so that conclusions can immediately be drawn for a particular element if such conditions can be checked.

This boundedness criterion is not new. It has indeed been indicated by the author since 1968[8] and no simpler one seems to have been presented. Unfortunately, in most cases, the boundedness conditions are really not easy to check and the patch test has to be used.

The last part of the paper is thus devoted to justify the patch test by proving that the boundedness conditions mentioned above are certainly satisfied by the approximate solution if they are also satisfied by the exact solution and if the patch test is passed. In other words, completeness and passing the patch test are proved to be sufficient conditions for convergence.

The whole discussion is valid for the general hybrid case which was not considered by any of the authors quoted above, except Kikushi[3]. It was however examined quite early by Pin-Tong[9] and more recently by Oden[10]. The author himself has been studying convergence of hybrid elements since 1970[11]. No general expression for the error was however given in his earlier papers, basically concerned with convergence and not with accuracy. Estimates of the order of approximation were indicated later[12] but the present paper is the first by the author to contain an explicit error expression.

In what concerns the patch test, its justification in the present paper makes it clear that: (i) passing the patch test is not a necessary condition for convergence; (ii) the simple patch test is sufficient for accuracy analysis.

This is in contradiction with what Strang and Fix[5] wrote in their book where (page 301) the simple patch test is expressly associated to simple convergence and the higher-order patch test seems to be declared necessary for accuracy analyses.

2. CONTINUOUS MODELS

Let us consider an elastic structure occupying a domain Ω with boundary β, subdivided into subdomains Ω^e with boundaries β^e. The external boundary β is assumed to consist of two parts, β_1 and β_2. Tractions are supposed to be prescribed on β_1 and displacements on β_2. The set of all the points belonging to the boundaries β^e, but not to the external boundary β, is termed the internal boundary γ of Ω.

Stress, strain and displacement fields are associated to the structure, strains and displacements being supposed so small that geometrical linearity can be admitted.

Stresses and strains are assumed connected by homogeneous (vanishing stresses correspond to vanishing strains) stress-strain equations

$$\mathbf{s} = \mathbf{H}\mathbf{e}, \tag{2.1}$$

strains and displacements by inhomogeneous (initial strains $\mathbf{e}^0$, associated with self-equilibrating initial stresses $\mathbf{s}^0$, correspond to rigid body displacements) linear strain-displacement equations

$$\mathbf{e} = \mathbf{D}\mathbf{u} + \mathbf{e}^0 \tag{2.2}$$

and stresses are assumed to satisfy the equilibrium equations

$$\mathbf{E}\mathbf{s} = \mathbf{f} \quad \text{in} \quad \Omega^e, \qquad \mathbf{N}\mathbf{s} = \mathbf{p} \quad \text{on} \quad \beta^e \tag{2.3-4}$$

where $\mathbf{f}$ and $\mathbf{p}$ respectively represent the body force density vector and the traction vector.

$\mathbf{H}$ is a positive-definite symmetric matrix. $\mathbf{D}$ and $\mathbf{E}$ are matrices of differential operators. $\mathbf{N}$ depends on the normal vector at each point on β^e and changes sign with the external normal vector, $\mathbf{n}$.

The strain and displacement fields are assumed continuous within each Ω^e (piece-wise continuous on Ω). The displacements are, in addition, assumed to admit within each Ω^e, the derivatives involved in the strain-displacement equations. The stresses, and therefore the strains, are assumed to admit, also within each Ω^e, the derivatives involved in the equilibrium equations. We call $\mathscr{E}$ and $\mathscr{U}$ the spaces of all the strains and displacements on Ω satisfying the conditions above.

Matrices $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{N}$ are supposed such that, for any vectors $\mathbf{s}$ and $\mathbf{e}$ satisfying such conditions, but not necessarily interconnected through (2.1) and (2.2), the identity

$$\int_{\Omega^e} \mathbf{s}^T(\mathbf{Du})\,d\Omega^e = \int_{\Omega^e} (\mathbf{Es})^T \mathbf{u}\,d\Omega^e + \int_{\beta^e} (\mathbf{Ns})^T \mathbf{u}\,d\beta^e \tag{2.5}$$

holds.

An immediate consequence of (2.5) is the work theorem for subdomain Ω^e,

$$\int_{\Omega^e} \mathbf{s}^T(\mathbf{e}-\mathbf{e}^0)\,d\Omega^e = \int_{\Omega^e} \mathbf{f}^T \mathbf{u}\,d\Omega^e + \int_{\beta^e} \mathbf{p}^T \mathbf{u}\,d\beta^e. \tag{2.6}$$

Each stress field, and thus each strain field, being equilibrated by a system of external forces distributed on the subdomains Ω^e, on β and on γ with densities respectively $\mathbf{f}$, $\mathbf{p}$ and $\mathbf{g}$, the work principle can take the global form

$$\sum_e \int_{\Omega^e} \mathbf{s}^T(\mathbf{e}-\mathbf{e}^0)\,d\Omega^e = \sum_e \int_{\Omega^e} \mathbf{f}^T \mathbf{u}\,d\Omega^e + \int_{\beta} \mathbf{p}^T \mathbf{u}\,d\beta + \int_{\gamma} \mathbf{g}^T \mathbf{u}\,d\gamma \tag{2.7}$$

where $\mathbf{s}$, $\mathbf{e}$ and $\mathbf{u}$ are supposed to satisfy the conditions above and $\mathbf{u}$ is assumed moreover to be continuous on the whole domain Ω.

The initial stresses being self-equilibrating, in the sense that they are equilibrated on Ω by vanishing external forces, we can write, by virtue of (2.7),

$$\sum_e \int_{\Omega^e} \mathbf{s}^{0T}(\mathbf{Du})\,d\Omega^e = 0 \tag{2.8}$$

for any $u \in \mathscr{U}$ continuous on Ω.

Equation (2.2) permits to generate, from a given displacement field $u \in \mathscr{U}$ and a given initial strain field $\mathbf{e}^0 \in \mathscr{E}$, a unique strain field $e \in \mathscr{E}$. A linear operator Δ may be introduced thus, with domain $\mathscr{U}$ and range $\mathscr{E}$, such that eqn (2.2) is equivalent to

$$e = \Delta u + e^0. \tag{2.9}$$

We call an isocompatible subset $\mathscr{U}_{\mathscr{C}}$ of $\mathscr{U}$ the set of all the displacement fields which take given values $\mathbf{q}$ on β_2 and present given discontinuities $\mathbf{h}$ on γ. We call an isocompatible subset of $\mathscr{E}$ the set $\mathscr{E}_{\mathscr{C},e^0}$ of all the strain fields generated by each of the elements of an isocompatible subset $\mathscr{U}_{\mathscr{C}}$ of $\mathscr{U}$ and a given initial strain field $\mathbf{e}^0$. Each element of such isocompatible subset is said to be compatibilized by a system of isocompatibilities characterized by an initial strain vector $\mathbf{e}^0$ on the subdomains Ω^e, displacement values $\mathbf{q}$ on β_2 and displacement discontinuities $\mathbf{h}$ on γ. We call $\Delta_{\mathscr{C},e^0}$ the subset of Δ with domain $\mathscr{U}_{\mathscr{C}}$ and range $\mathscr{E}_{\mathscr{C},e^0}$.

A unique displacement field $u \in \mathscr{U}$ (and also a unique initial strain field $e^0 \in \mathscr{E}$) is associated to a given strain field e if, as we assume, conditions are given on β_2 and γ for supporting the structure and interconnecting its different parts in such a way that no rigid body displacements of such parts are allowed. This means that operator $\Delta_{\mathscr{C},e^0}$ has an inverse, $\Phi_{\mathscr{C},e^0}$, such that

$$u = \Phi_{\mathscr{C},e^0} e \quad \text{if} \quad e \in \mathscr{E}_{\mathscr{C},e^0}. \tag{2.10}$$

Introducing (2.10) in (2.9) we conclude that operators $\Phi_{\mathscr{C},e^0}$ and Δ are such that, for any $e \in \mathscr{E}_{\mathscr{C},e^0}$,

$$e = \Delta\Phi_{\mathscr{C},e^0} e + e^0. \tag{2.11}$$

Let $\mathscr{F}$ be the space of all the systems of external forces which equilibrate the elements of $\mathscr{E}$. Each system, characterized by densities $\mathbf{f}$, $\mathbf{p}$ and $\mathbf{g}$, is equilibrated by reactions distributed on β_2. Two systems are admitted to coincide if such reactions are statically equivalent.

Let Π be an operator with domain $\mathscr{E}$ and range $\mathscr{F}$ which associates to each strain field $e \in \mathscr{E}$ the element $f \in \mathscr{F}$ which equilibrates such field. The equation

$$\Pi e = f \tag{2.12}$$

represents the equilibrium equations. Operator Π has no inverse, unless the structure is statically determinate. The subset of all the elements in $\mathscr{E}$ which correspond to a given element of $\mathscr{F}$ is called an isoequilibrated subset of $\mathscr{E}$.

We call $\Pi_{\mathscr{C},e^0}$ the subset of Π which associates to each element of an isocompatible subset $\mathscr{E}_{\mathscr{C},e^0} \subset \mathscr{E}$ the corresponding element of $\mathscr{F}$. The intersection between each isocompatible and each isoequilibrated subset of $\mathscr{E}$ being assumed unique (uniqueness principle), operator $\Pi_{\mathscr{C},e^0}$ admits an inverse,

$$\mathrm{P}_{\mathscr{C},e^0} = \Pi^{-1}_{\mathscr{C},e^0}. \tag{2.13}$$

A norm may be associated to each element of spaces $\mathscr{E}$, $\mathscr{U}$ and $\mathscr{F}$, which become thus Banach spaces. We choose

$$\|e\|_{\mathscr{E}} = \sqrt{\left(\sum_e \int_{\Omega^e} \mathbf{e}^T \mathbf{e}\, d\Omega^e\right)} \tag{2.14}$$

$$\|u\|_{\mathscr{U}} = \sqrt{\left(\sum_e \int_{\Omega^e} \mathbf{u}^T \mathbf{u}\, d\Omega^e\right)} \tag{2.15}$$

and finally

$$\|f\|_{\mathscr{F}} = \sqrt{\left(\sum_e \int_{\Omega^e} \mathbf{f}^T \mathbf{f}\, d\Omega^e + \int_{\beta_1} \mathbf{p}^T \mathbf{p}\, d\beta + \int_{\gamma} \mathbf{g}^T \mathbf{g}\, d\gamma\right)}. \tag{2.16}$$

The definition of these norms gives a meaning to the boundedness of operators Δ, $\Phi_{\mathscr{C},e^0}$, Π and $\mathrm{P}_{\mathscr{C},e^0}$, which will be admitted from now on.

Figure 1 may help in keeping in mind all the spaces and operators which have been introduced up to now and also those which will be introduced in the sequel.

3. HYBRID DISCRETE MODELS

Let the structure be discretized into finite elements corresponding to the subdomains Ω^e. A discrete model is thus generated which is analogous to the generating continuous one.

The discrete domain is the set of the subdomains Ω^e. The continuous fields are replaced by sets of generalized stress, strain and displacement vectors, $\mathbf{s}$, $\mathbf{e}$ and $\mathbf{u}$, associated to the different subdomains. Discrete stress-strain, strain-displacement and equilibrium equations,

$$\mathbf{s}^e = \mathbf{H}^e \mathbf{e}^e \tag{3.1}$$

$$\mathbf{e}^e = \mathbf{D}^e \mathbf{u}^e + \mathbf{e}^{0e} \tag{3.2}$$

$$\mathbf{E}^e \mathbf{s}^e = \mathbf{f}^e \tag{3.3}$$

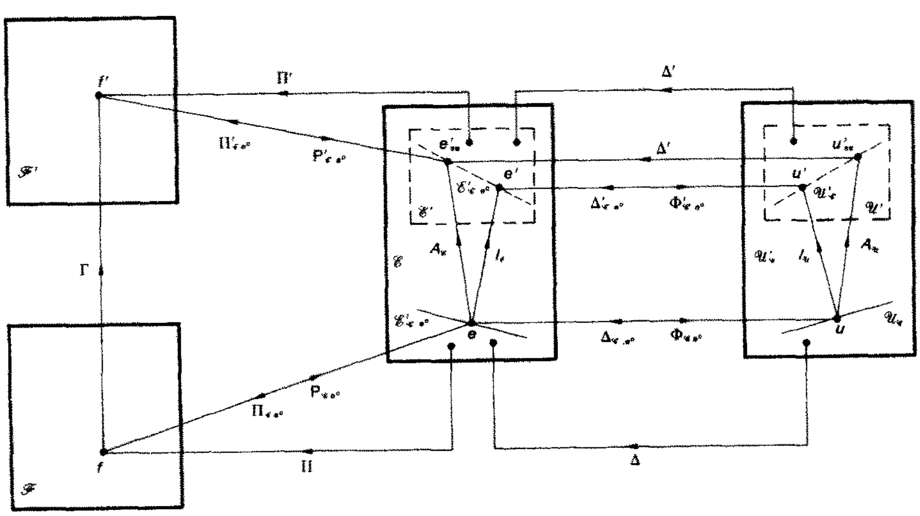

Fig. 1.

are associated to each Ω^e. Equation (2.5) becomes

$$\mathbf{s}^{eT}\mathbf{D}^e\mathbf{u}^e = (\mathbf{E}^e\mathbf{s}^e)^T\mathbf{u}^e \tag{3.4}$$

and, as it must be valid for any vectors $\mathbf{s}^e$ and $\mathbf{u}^e$ not necessarily interconnected through eqns (3.1)–(3.3) it is equivalent to

$$\mathbf{D}^e = \mathbf{E}^{eT}. \tag{3.5}$$

Equations (2.6) and (2.7) respectively become

$$\mathbf{s}^{eT}(\mathbf{e}^e - \mathbf{e}^{0e}) = \mathbf{f}^{eT}\mathbf{u}^e \tag{3.6}$$

and

$$\sum_e \mathbf{s}^{eT}(\mathbf{e}^e - \mathbf{e}^{0e}) = \mathbf{F}^T\mathbf{U} \tag{3.7}$$

where $\mathbf{F}$ and $\mathbf{U}$ denote the global force and displacement vectors.

The discrete self-equilibrium condition for $\mathbf{s}^{0e}$ is now

$$\sum_e \mathbf{s}^{0eT}\mathbf{D}^e\mathbf{u}^e = 0 \tag{3.8}$$

where

$$\mathbf{u}^e = \mathbf{A}^e\mathbf{U}. \tag{3.9}$$

Equation (3.9) is the expression of the discrete continuity conditions for displacements. $\mathbf{A}^e$ is the connectivity matrix associated with subdomain Ω^e.

Usually, the elements of $\mathbf{u}^e$ are generalized displacements at certain given points on β^e called nodal points or nodes, and the discrete continuity conditions (3.9) require the continuity of the displacements at such points.

Assuming the discrete model to be generated by the potential energy method[11], continuous and discrete strains and displacements are respectively interconnected by equations

$$\mathbf{e} = \chi^e\mathbf{e}^e \quad \text{and} \quad \mathbf{u} = \varphi^e\mathbf{u}^e \tag{3.10-11}$$

which define the continuous fields allowed within Ω^e.

We denote by $\mathscr{E}'$ and $\mathscr{U}'$ the subspaces of $\mathscr{E}$ and $\mathscr{U}$ containing all the strain and displacement fields which respectively satisfy (3.10) and (3.11). Isocompatible subsets with respect to discrete compatibility conditions can of course be defined on $\mathscr{E}'$ and $\mathscr{U}'$, in the same way as on $\mathscr{E}$ and $\mathscr{U}$ with respect to the continuous ones. Operators Δ' and $\Phi'_{\mathscr{E},e^{0'}}$ can be introduced on the other hand as well as equation

$$e' = \Delta' u' + e^{0'}. \tag{3.12}$$

Three principles are admitted which make it possible to express the discrete magnitudes in terms of the continuous ones.

The first of these is the principle of the invariance of internal work, which implies

$$\mathbf{e}_1{}^{eT}\mathbf{H}^e\mathbf{e}_2{}^e = \int_{\Omega^e} \mathbf{e}_1{}^T\mathbf{H}\mathbf{e}_2\, d\Omega^e \tag{3.13}$$

for any pair of strain fields e_1 and e_2 belonging to $\mathscr{E}'$. Introducing (3.10), we obtain

$$\mathbf{H}^e = \int_{\Omega^e} \chi^{eT}\mathbf{H}\chi^e\, d\Omega^e. \tag{3.14}$$

From (3.13) we can obtain also

$$\mathbf{e}^e = \mathbf{H}^{e^{-1}} \int_{\Omega^e} \chi^{e^T} \mathbf{H} e \, d\Omega^e \tag{3.15}$$

i.e. the expression (valid only if $e \in \mathscr{E}'$) of the discrete strain vector in terms of the continuous one.

Expression (3.15) may be used for expressing the discrete initial strains in terms of the continuous ones, if these last are assumed to be allowed within the element ($e^0 \in \mathscr{E}'$), as we admit them to be. We write therefore

$$\mathbf{e}^{0^e} = \mathbf{H}^{e^{-1}} \int_{\Omega^e} \chi^{e^T} \mathbf{H} e^0 \, d\Omega^e. \tag{3.16}$$

The assumption of e^0 belonging to $\mathscr{E}'$ should not surprise. It is well-known indeed that the remaining incompatibilities, i.e. the displacements prescribed on β_2 and the displacement discontinuities prescribed on γ, cannot be arbitrary also, as they must comply with the displacement fields allowed within the elements.

The second principle requires the discrete compatibility conditions to be a subset of the continuous ones. This implies that

$$e = \Delta' u + e^0 \quad \text{if (2.9) holds,} \quad e \in \mathscr{E}' \quad \text{and} \quad u \in \mathscr{U}', \tag{3.17}$$

or that (3.2) is satisfied if (2.2) is also satisfied. Introducing (2.2) and then (3.11) in (3.15), we obtain

$$\mathbf{e}^e = \mathbf{H}^{e^{-1}} \int_{\Omega^e} \chi^{e^T} \mathbf{H}(\mathbf{D}\varphi^e) \, d\Omega^e \mathbf{u}^e + \mathbf{H}^{e^{-1}} \int_{\Omega^e} \chi^{e^T} \mathbf{H} e^0 \, d\Omega^e \tag{3.18}$$

and then, comparing with (3.2) and considering that the second term in the right-hand side is e^{0^e},

$$\mathbf{D}^e = \mathbf{H}^{e^{-1}} \int_{\Omega^e} \chi^{e^T} \mathbf{H}(\mathbf{D}\varphi^e) \, d\Omega^e \tag{3.19}$$

or, by virtue of eqn (2.5), the more usual[7] expression

$$\mathbf{D}^e = \mathbf{H}^{e^{-1}} \left[\int_{\Omega^e} (\mathbf{E}\mathbf{H}\chi^e)^T \varphi^e \, d\Omega^e + \int_{\beta^e} (\mathbf{N}\mathbf{H}\chi^e)^T \varphi^e \, d\beta^e \right]. \tag{3.20}$$

It is now easy to prove that the discrete initial stresses $\mathbf{s}^{0^e}$ corresponding to the initial strains defined by (3.16) are self-equilibrating in the discrete sense. It suffices to show indeed that (3.8) holds if (2.8) also holds, and this can be done by introducing (3.10) and (3.11) in (2.8). We obtain

$$\sum_e \mathbf{e}^{0^{eT}} \int_{\Omega^e} \chi^{e^T} \mathbf{H}(\mathbf{D}\varphi^e) \, d\Omega^e \mathbf{u}^e = 0 \tag{3.21}$$

and, therefore, by virtue of (3.19), eqn (3.8).

The third principle is the principle of the invariance of external work which, applied to each element, implies

$$\mathbf{f}^{e^T} \mathbf{u}^e = \int_{\Omega^e} \mathbf{f}^T \mathbf{u} \, d\Omega^e + \int_{\beta^e} \mathbf{p}^T \mathbf{u} \, d\beta^e \tag{3.22}$$

for any $u \in \mathscr{U}'$, and thus, introducing (3.11),

$$\mathbf{f}^e = \int_{\Omega^e} \varphi^{e^T} \mathbf{f} \, d\Omega^e + \int_{\beta^e} \varphi^{e^T} \mathbf{p} \, d\beta^e. \tag{3.23}$$

Applied to the whole structure, the third principle gives

$$\mathbf{F}=\sum_e A^{eT}\Big(\int_{\Omega^e}\varphi^{eT}\,\mathbf{f}\,d\Omega^e+\int_{\beta^e\cap\beta_1}\varphi^{eT}\,\mathbf{p}\,d\beta^e+\frac{1}{2}\int_{\beta^e\cap\gamma}\varphi^{eT}\,\mathbf{g}\,d\gamma\Big) \tag{3.24}$$

where factor 1/2 multiplying the last term results from assuming that the external forces distributed on the internal boundary are equally shared by each of the two elements contacting at each point of γ.

Two interpolation operators (bounded and linear) are introduced which respectively associate to each element of $\mathcal{U}$ or $\mathcal{E}$ an element of $\mathcal{U}'$ or $\mathcal{E}'$.

Let $I_{\mathcal{U}}$ be the interpolation operator for displacements, with domain $\mathcal{U}$ and range $\mathcal{U}'$, such that the $I_{\mathcal{U}}$-image of an element $u\in\mathcal{U}$ is the element $u'\in\mathcal{U}'$ whose components take values equal to those of u at the nodes. We write

$$u'=I_{\mathcal{U}}u \tag{3.25}$$

and point out that

$$u=I_{\mathcal{U}}u \quad \text{if} \quad u\in\mathcal{U}'. \tag{3.26}$$

The interpolation operator for strains, $I_{\mathcal{E}}$, with domain $\mathcal{E}$ and range $\mathcal{E}'$, is defined by

$$I_{\mathcal{E}}e=\Delta' I_{\mathcal{U}}\Phi_{\mathcal{E},e^0}e+e^0 \quad \text{if} \quad e\in\mathcal{E}_{\mathcal{E},e^0} \tag{3.27}$$

which implies

$$e^0=I_{\mathcal{E}}e^0 \tag{3.28}$$

because $\Phi_{\mathcal{E},e^0}e^0=0$.

Introducing (2.10) and (3.28) in (3.27), we conclude that

$$I_{\mathcal{E}}(e-e^0)=\Delta' I_{\mathcal{U}}u \quad \text{if} \quad e-e^0=\Delta u. \tag{3.29}$$

It may be proved also that

$$e=I_{\mathcal{E}}e \quad \text{if} \quad e\in\mathcal{E}' \quad \text{and} \quad \Phi_{\mathcal{E},e^0}e\in\mathcal{U}'. \tag{3.30}$$

Indeed, by virtue of (3.26),

$$I_{\mathcal{E}}e=\Delta' u+e^0 \tag{3.31}$$

and, thus, comparison with (3.12) immediately yields (3.30).

As each isocompatible subset of $\mathcal{U}'$ corresponds to given displacement values at the nodes located on β_2 and to given displacement discontinuities at nodes located on γ, the $I_{\mathcal{U}}$-images of fields isocompatible in $\mathcal{U}$ are isocompatible in $\mathcal{U}'$. Similarly, the $I_{\mathcal{E}}$-images of isocompatible fields in $\mathcal{E}$ are also incompatible in $\mathcal{E}'$.

The isocompatible subset $\mathcal{U}'_{\mathcal{E}}$ of $\mathcal{U}'$ which contains the $I_{\mathcal{U}}$-images of the elements of a given isocompatible subset $\mathcal{U}_{\mathcal{E}}$ of $\mathcal{U}$ is said to correspond to $\mathcal{U}_{\mathcal{E}}$. A similar definition may be given for a isocompatible subset of $\mathcal{E}'$ corresponding to a given isocompatible subset of $\mathcal{E}$.

To each set of vectors $\mathbf{f}^e$ (one for element) a system f' of discrete external forces correspond. We call $\mathcal{F}'$ the space of all such systems, i.e. the discrete counterpart of $\mathcal{F}$. Each element of $\mathcal{F}'$ is characterized by a vector $\mathbf{F}_1$ of discrete external forces acting at the nodes not located on β_2. Such vector is connected to the vector fields $\mathbf{f}$, $\mathbf{p}$ and $\mathbf{g}$, which characterize the continuous external force system to which it corresponds, by the equation

$$\mathbf{F}_1=\sum_e \mathbf{A}_1^{eT}\Big(\int_{\Omega^e}\varphi^{eT}\,\mathbf{f}\,d\Omega^e+\int_{\beta^e\cap\beta_1}\varphi^{eT}\,\mathbf{p}\,d\beta^e+\frac{1}{2}\int_{\beta^e\cap\gamma}\varphi^{e}\,\mathbf{g}\,d\beta^e\Big) \tag{3.32}$$

which results from (3.24). $\mathbf{A}_1^e$ is the connectivity matrix associated with Ω^e and the nodes not located on β_2.

The norm in $\mathcal{F}'$ is defined by

$$\|f'\|_{\mathcal{F}'} = \sqrt{(\mathbf{F}_1^T\mathbf{F}_1)}. \tag{3.33}$$

We introduce Π', $\Pi'_{\mathscr{C}',e^{0'}}$ and $P'_{\mathscr{C}',e^{0'}}$ as the discrete counterparts of Π, $\Pi_{\mathscr{C},e^0}$ and $P_{\mathscr{C},e^0}$.

Operator Γ, with domain $\mathcal{F}$ and range $\mathcal{F}'$ is such that equation

$$f' = \Gamma f \tag{3.34}$$

represents (3.32).

All these operators are supposed bounded.

The set of elements $e' \in \mathscr{E}'$ which satisfy the equation

$$\Pi' e' = f' \tag{3.35}$$

for a given $f' \in \mathcal{F}'$ is an isoequilibrated subset of $\mathscr{E}'$. The intersection between each isocompatible and each isoequilibrated subset of $\mathscr{E}'$ is assumed unique.

An isoequilibrated subset $\mathscr{D}'$ of $\mathscr{E}'$ is said to correspond to an isoequilibrated subset $\mathscr{D}$ of $\mathscr{E}$ associated to an element f of $\mathcal{F}$ if $\mathscr{D}'$ is associated to Γf.

We finally introduce the bounded but generally non-linear approximation operators $A_{\mathscr{E}}$ and $A_{\mathscr{U}}$ with domains respectively $\mathscr{E}$ and $\mathscr{U}$ and ranges $\mathscr{E}'$ and $\mathscr{U}'$.

Operator $A_{\mathscr{E}}$ associates to the intersection $e \in \mathscr{E}$ of given isocompatible and isoequilibrated subsets of $\mathscr{E}$ the intersection $e'_a \in \mathscr{E}'$ of the corresponding subsets of $\mathscr{E}'$. We write

$$e'_a = A_{\mathscr{E}} e \tag{3.36}$$

and call e'_a the approximation of e in $\mathscr{E}'$. This definition makes it clear that e'_a and e' (the $I_{\mathscr{E}}$-image of e) are isocompatible in $\mathscr{E}'$.

It is clear also that

$$e^0 = A_{\mathscr{E}} e^0. \tag{3.37}$$

This results from (3.28) and from e^0 being selfequilibrating both in the discrete and in the continuous sense, and thus simultaneously belonging to an isoequilibrated subset of $\mathscr{E}$ and to the corresponding isoequilibrated subset of $\mathscr{E}'$.

Operator $A_{\mathscr{U}}$ associates to the $\Phi_{\mathscr{C},0}$-image of an element $e \in \mathscr{E}_{\mathscr{C},e^0}$ the $\Phi'_{\mathscr{C},0}$-image of its approximation in $\mathscr{E}'$. We write

$$u'_a = A_{\mathscr{U}} u \tag{3.38}$$

or

$$u'_a = \Phi'_{\mathscr{C},0} A_{\mathscr{E}} \Delta u \tag{3.39}$$

if $u \in \mathscr{U}_{\mathscr{C}}$ and $\mathscr{U}'_{\mathscr{C}}$ corresponds to $\mathscr{U}_{\mathscr{C}}$.

Introducing (2.9) and considering that $\Phi'_{\mathscr{C}',e^{0'}}$, is the inverse of $\Delta'_{\mathscr{C}',e^{0'}}$, we obtain

$$A_{\mathscr{E}}(e - e^0) = \Delta'(A_{\mathscr{U}} u). \tag{3.40}$$

In the non-hybrid case, operator Δ' is a subset of Δ and we can write

$$A_{\mathscr{U}}(e - e^0) - (e - e^0) = \Delta(A_{\mathscr{U}} u - u), \tag{3.41}$$

equation which may be used for connecting the orders of magnitude of the strain and displacement approximation errors.

4. THE APPROXIMATION THEOREM

An inequality will be established now with the help of which upper bounds for the strain approximation error can be determined.

The derivation of such inequality is based on the well-known total potential energy (t.p.e.) theorem.

In the continuous model, the t.p.e. associated with $f \in \mathscr{F}$ is the functional on $\mathscr{E} \times \mathscr{U}$

$$T_f(e, u) = \sum_e U^e(e) - (f, u) \tag{4.1}$$

where $U^e(e)$ is the strain energy associated with the subdomain Ω^e and the strain field e, and

$$(f, u) = \sum_e \int_{\Omega^e} \mathbf{f}^T \mathbf{u} \, d\Omega^e + \int_{\beta_1} \mathbf{p}^T \mathbf{u} \, d\beta + \int_\gamma \mathbf{g}^T \mathbf{u} \, d\gamma \tag{4.2}$$

In the discrete model, the t.p.e. takes the form

$$T'_{f'}(e', u') = \sum_e U^e(e') - (f', u')' \tag{4.3}$$

where

$$(f', u')' = \sum_e \mathbf{F}_1{}^T \mathscr{U}_1. \tag{4.4}$$

By virtue of the invariance of work (see Section 3),

$$T'_{f'}(e, u) = T_f(e, u) \quad \text{if} \quad e \in \mathscr{E}', \quad u \in \mathscr{U}', \quad f' = \Gamma f. \tag{4.5}$$

The t.p.e. theorem for the continuous model states that the continuous (or exact) solution, i.e. $(e_s, u_s) \in \mathscr{E} \times \mathscr{U}$, where e_s is the solution of the equation

$$\Pi_{\mathscr{C}, e^0} e = f \tag{4.6}$$

and

$$u_s = \Phi_{\mathscr{C}, e^0} e_s, \tag{4.7}$$

minimizes $T_f(e, u)$ on $\mathscr{E}_{\mathscr{C}, e^0} \times \mathscr{U}_{\mathscr{C}}$.

The t.p.e. theorem for the discrete model states that the discrete (or approximate) solution, i.e. $(e'_{sa}, u'_{sa}) \in \mathscr{E}' \times \mathscr{U}'$, where e'_{sa} is the solution of the equation

$$\Pi'_{\mathscr{C}', e^{0'}} e' = f' \tag{4.8}$$

and

$$u'_{sa} = \Phi'_{\mathscr{C}', e^{0'}} e'_{sa}, \tag{4.9}$$

minimizes $T'_{f'}(e', u')$ on $\mathscr{E}'_{\mathscr{C}', e^{0'}} \times \mathscr{U}'_{\mathscr{C}'}$.

Both theorems suppose stability.

The main problem which we wish to discuss in the present paper is the approximation of e_s by e'_{sa}.

An essential point in any approximation analysis being the definition of the distance between two fields in $\mathscr{E}$, we define such distance as the square root of the strain energy of their difference, i.e.

$$d(e_1, e_2) = \sqrt{\left(\sum_e U^e(e_2 - e_1)\right)}. \tag{4.10}$$

It is well-known that, if this definition is adopted, the distance between the strain field associated to the exact solution $e_s \in \mathscr{E}_{\mathscr{C},e^0}$ and any field e_c isocompatible with e_s, i.e. also belonging to $\mathscr{E}_{\mathscr{C},e^0}$, satisfies

$$d(e_s, e_c) = \sqrt{(T_f(u_c, e_c) - T_f(u_s, e_s))}. \tag{4.11}$$

Definition (4.10) and eqn (4.11) are in principle valid only for the linear case. In the non-linear case, both can still be used only for points very near each other. The approximation theorem which is going to be established can thus be applied also in the non-linear case for evaluating the order of the magnitude of the distance between the exact and approximate solutions, provided such distance really tends to zero.

Consider now fields e_{sa} and u_{sa}, respectively belonging to $\mathscr{E}_{\mathscr{C},e^0}$ and $\mathscr{U}_{\mathscr{C}}$ and such that

$$e'_{sa} = I_{\mathscr{E}} e_{sa}, \quad u'_{sa} = I_{\mathscr{U}} u_{sa}, \tag{4.12-13}$$

and let

$$e'_s = I_{\mathscr{E}} e_s, \quad u'_s = I_{\mathscr{U}} u_s. \tag{4.14-15}$$

Field u'_s belongs to $\mathscr{U}'_{\mathscr{C}}$ and e'_s belongs to $\mathscr{E}'_{\mathscr{C},e^{0'}}$.

Let $f' = \Gamma f$ and

$$\delta_s T = T'_{f'}(e'_s, u'_s) - T_f(e_s, u_s) \tag{4.16}$$

$$\delta_a T = T'_{f'}(e'_{sa}, u'_{sa}) - T_f(e_{sa}, u_{sa}) \tag{4.17}$$

By virtue of the invariance of external work,

$$\delta T = T'_{f'}(I_{\mathscr{E}} e, I_{\mathscr{U}} u) - T_f(e, u) = \sum_e \delta U^e - (f, \delta u) \tag{4.18}$$

where

$$\delta e = e' - e, \quad \delta u = u' - u \tag{4.19-20}$$

and

$$\delta U^e = U^e(e') - U^e(e). \tag{4.21}$$

Before considering the approximation theorem, we can show that

$$\delta T = 0(\|\delta e\|_{\mathscr{E}}) + 0(\|\delta u\|_{\mathscr{U}}). \tag{4.22}$$

Indeed, if s is the stress field corresponding to e,

$$\sum_e \delta U^e = \sum_e \int_{\Omega^e} \mathbf{s}^T \delta \mathbf{e} \, d\Omega^e \leq \sqrt{\left(\sum_e \int_{\Omega^e} \mathbf{s}^T \mathbf{s} \, d\Omega^e \sum_e \int_{\Omega^e} \delta \mathbf{e}^T \delta \mathbf{e} \, d\Omega^e\right)} = \|s\|_{\mathscr{S}} \|\delta e\|_{\mathscr{E}}. \tag{4.23}$$

On the other hand,

$$(f, \delta u) \leq \|f\|_{\mathscr{F}} \|\delta u\|_{\mathscr{U}}, \tag{4.24}$$

so that we may write

$$\delta T \leq \|s\|_{\mathscr{S}} \|\delta e\|_{\mathscr{E}} + \|f\|_{\mathscr{F}} \|\delta u\|_{\mathscr{U}}, \tag{4.25}$$

where

$$\|s\|_{\mathscr{S}} = \sqrt{\left(\sum_e \int_{\Omega^e} s^T s \, d\Omega^e\right)}. \quad (4.26)$$

On the other hand,

$$d(e, e') = \sqrt{\left(\sum_e U^e(\delta e)\right)} = \sqrt{\left(\frac{1}{2}\sum_e \int_{\Omega^e} \delta s^T \delta e \, d\Omega^e\right)} = 0(\|\delta e\|_{\mathscr{E}}) \quad (4.27)$$

so that

$$d(e_{sa}, e'_{sa}) = 0(\|\delta_a e\|_{\mathscr{E}}). \quad (4.28)$$

The approximation theorem [12] simply states that:

"If the distance between two fields in $\mathscr{E}$ is defined by (4.10), then, the distance $d(e_s, e'_{sa})$ between the exact and the approximate solutions satisfies the inequality

$$d(e_s, e'_{sa}) \leq \sqrt{(|\delta_s T| + |\delta_a T|)} + 0(\|\delta_a e\|_{\mathscr{E}})". \quad (4.29)$$

The proof is straightforward and is schematized in Fig. 2.

5. ERROR BOUNDS

Let δu and δe be defined by (4.19) and (4.20) and let ℓ^e and ℓ respectively denote the diameter of Ω^e and the maximum value of ℓ^e on the whole set of subdomains.

The following theorems establish bounds for $\|\delta u\|_{\mathscr{U}}$ and $\|\delta e\|_{\mathscr{E}}$.

Theorem 5.1: "If (a) the displacement field $u \in \mathscr{U}$ is such that the $(n+1)$th derivatives of its components are all bounded within each subdomain Ω^e, their moduli being, at each point, lower that a positive number U_{n+1}; (b) each polynomial displacement field of the nth degree (or less) is allowed within Ω^e; then

$$\|\delta u\|_{\mathscr{U}} \leq \frac{(1 + \|I_{\mathscr{U}}\|)U_{n+1}}{(n+1)!} l^{n+1} \sqrt{(\Omega)}". \quad (5.1)$$

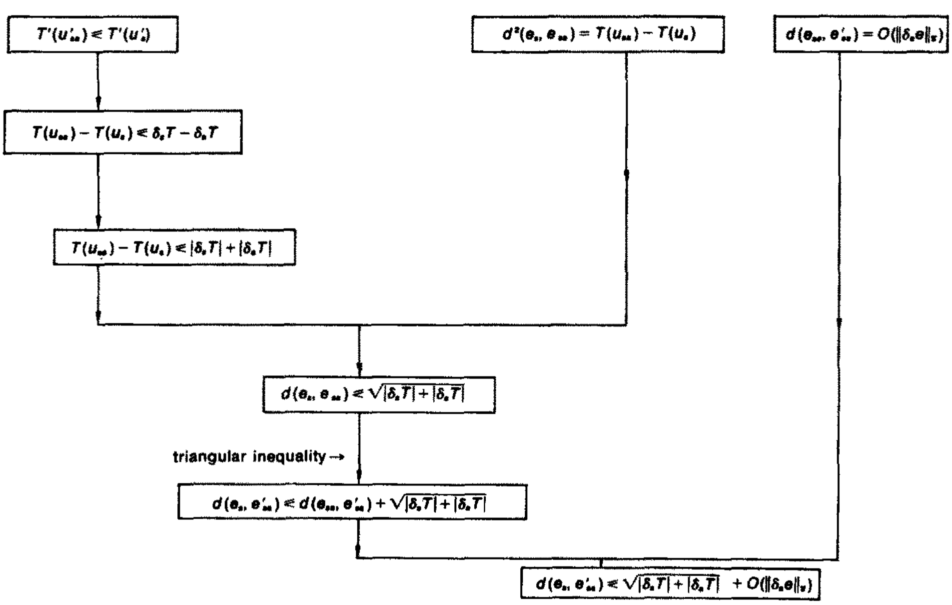

Fig. 2.

In order to prove this theorem, it suffices to remark that component u_i can be expressed within Ω^e by

$$u_i = u_{ni}(\mathbf{O}) + \frac{1}{(n+1)!} u^{(n+1)}_{i,jk\ldots l}(\mathbf{O}_i)(x_j - x_j^0)(x_k - x_k^0)\ldots(x_\ell - x_\ell^0) \tag{5.2}$$

where $u_{ni}(\mathbf{x})$ represents a polynominal of the nth degree and $\mathbf{O}$ and $\mathbf{O}_i$ are points within Ω^e
We can write therefore

$$u = u_n + r_{n+1} \tag{5.3}$$

where u_n represents the field with components u_{ni} and

$$\|r_{n+1}\|_{\mathscr{U}} \leq \frac{U_{n+1}}{(n+1)!} \ell^{n+1}\sqrt{\Omega}. \tag{5.4}$$

As, by virtue of assumption (b),

$$u_n \in \mathscr{U}' \tag{5.5}$$

we can write, by virtue of (3.26),

$$I_{\mathscr{U}}u - u_n = I_u(u - u_n) \tag{5.6}$$

and, therefore,

$$\|I_{\mathscr{U}}u - u_n\|_{\mathscr{U}} \leq \|I_{\mathscr{U}}\|\|u - u_n\|_{\mathscr{U}}. \tag{5.7}$$

Thus, by virtue of (5.3) and (5.4),

$$\|I_{\mathscr{U}}u - u_n\|_{\mathscr{U}} \leq \|I_{\mathscr{U}}\| \frac{U_{n+1}}{(n+1)!} \ell^{n+1}\sqrt{\Omega}. \tag{5.8}$$

As, on the other hand,

$$\|I_{\mathscr{U}}u - u\|_{\mathscr{U}} \leq \|I_{\mathscr{U}}u - u_n\|_{\mathscr{U}} + \|u_n - u\|_{\mathscr{U}}, \tag{5.9}$$

inequality (5.1) is proved.

Theorem 5.2: "If (a) the strain field $e \in \mathscr{E}$ is such that the $(m+1)$th derivatives of its components are all bounded within each subdomain Ω^e, their moduli being, at each point, lower than a positive number E_{m+1}; (b) all polynomial displacement fields of the $(m+d)$th degree (or less) are allowed within Ω^e, d being the order of the derivatives involved in the strain-displacement equations; (c) any strain field is also allowed which can be generated by such displacement fields with the help of eqn (2.9); then,

$$\|\delta e\|_{\mathscr{E}} \leq \frac{(1 + \|I_{\mathscr{E}}\|)E_{m+1}}{(m+1)!} \ell^{m+1}\sqrt{(\Omega)}". \tag{5.10}$$

Indeed, making $n = m + d$, eqn (5.3) yields

$$e = \Delta u + e^0 = \Delta u_{m+d} + e^0 + \Delta r_{m+d+1}. \tag{5.11}$$

As, by virtue of (b) and (c), u_{m+d} belongs to $\mathscr{U}'$ and

$$e_m = \Delta u_{m+e} + e^0 \tag{5.12}$$

belongs to $\mathscr{E}'$, there results that, by virtue of (3.30),

$$e_m = I_{\mathscr{E}} e_m. \tag{5.13}$$

We can write therefore

$$\|I_{\mathscr{E}} e - e_m\|_{\mathscr{E}} \leq \|I_{\mathscr{E}}\| \|e - e_m\|_{\mathscr{E}}. \tag{5.14}$$

Now, by analogy with (5.4),

$$\|e - e_m\|_{\mathscr{E}} = \|\Delta r_{m+d+1}\|_{\mathscr{E}} \leq \frac{E_{m+1}}{(m+1)!} \ell^{m+1} \sqrt{\Omega} \tag{5.15}$$

so that the use of the triangular inequality leads to (5.10).

Introducing (5.1) and (5.10) in (4.25), and then in (4.29), there results

$$\mathrm{d}(e_s, e'_{sa}) \leq \sqrt{\left(\left\{\frac{1+\|I_{\mathscr{E}}\|}{(m+1)!} [(\|s\|_{\mathscr{S}} E_{m+1})_s + (\|s\|_{\mathscr{S}} E_{m+1})_a] \ell^{m+1} + \frac{1+\|I_{\mathscr{U}}\|}{(n+1)!} \|f\|_{\mathscr{F}} [(U_{n+1})_s + (U_{n+1})_a] \ell^{n+1}\right\}\right)} \tag{5.16}$$

where indices s and a refer to the isocompatible fields e_s and e_{sa}.

An interesting point about eqn (5.16) is that n is not necessarily equal to $m + d$. The way in which the displacements are discretized, intervenes indeed in two distinct discretization steps: the determination of matrix $\mathbf{D}^e$ and the determination of the discrete external force vector $\mathbf{f}^e$. Different matrices φ^e may then be used for each, i.e. we can make

$$\mathbf{D}^e = \mathbf{H}^{e^{-1}} \int_{\Omega^e} \chi^{e^T} \mathbf{H}(\mathbf{D}\varphi_1{}^e)\, \mathrm{d}\Omega^e \tag{5.17}$$

and

$$\mathbf{f}^e = \int_{\Omega^e} \varphi_2{}^{e^T} \mathbf{f}\, \mathrm{d}\Omega^e + \int_{\beta^e} \varphi_2{}^{e^T} \mathbf{p}\, \mathrm{d}\beta_e. \tag{5.18}$$

Now, while the displacement discretization connected with $\varphi_1{}^e$ must be such that $n = m + d$, the value of n connected with $\varphi_2{}^e$ does not depend on the value of m.

We can write therefore

$$\mathrm{d}^2(e_s, e'_{sa}) = O(\ell^{m+1}) + O(\ell^{n+1}) \tag{5.19}$$

and assume that n refers to $\varphi_2{}^e$, i.e. to the displacement discretization matrix which is used for the determination of the discrete external forces.

In order that the approximation in strains be consistent with the approximation in displacements, m must be equal to n. If n is indeed larger than m (equal to $m + d$, for instance, as often people think it must be), the approximation in displacements is unnecessarily high. If m is larger than n, the waste is associated with the strains.

This conclusion must be present when the discrete forces are evaluated. The use of simple devices for the determination of such forces, like the one consisting in assigning to each node an equal share of the total forces acting in the element, which can be shown to be consistent with the use of constant allowed strains, can namely be justified with its help.

The following theorem results from the approximation theorem and from Theorems 5.1 and 5.2.

Theorem 5.3: "If (a) the moduli of the derivatives of order respectively $n + 1$ and $m + 1$ of the displacements and strains associated with the exact solution are lower than positive numbers

$(U_{n+1})_s$ and $(E_{m+1})_s$ within each subdomain Ω^e; (b) fields u_{sa} and e_{sa} exist, isocompatible with fields u_s and e_s, such that eqns (4.12) and (4.13) hold and the moduli of the derivatives of order $n+1$ and $m+1$ of their respective components are lower than positive numbers $(U_{n+1})_a$ and $(E_{m+1})_a$ within each subdomain Ω^e; (c) all polynomial displacement fields of the $(m+d)$th degree (or less) and all strain fields which can be generated by such displacement fields are allowed within each element, for the purpose of the determination of matrices $\mathbf{H}^e$ and $\mathbf{D}^e$; (d) all polynomial displacement fields of the nth degree (or less) are allowed within each element, for the purpose of the determination of the discrete external forces; then, the distance between the exact and the approximate solutions satisfies inequality (5.16)."

Conditions (c) and (d) are called completeness conditions.

In case of conformity (Ritz's case), $n = m + d$, and conditions (a), (c) and (d) are sufficient for the distance satisfying the inequality

$$\mathrm{d}(e_s, e'_{sa}) \leqslant \sqrt{\left(\frac{1+\|I_{\mathscr{E}}\|}{(m+1)!}(\|s\|_{\mathscr{E}} E_{m+1})_s \ell^{m+1}\right)} \tag{5.20}$$

and thus being of the order of $\ell^{(m+1)/2}$. In the general non-conforming case, however, the condition of the boundedness of the derivatives associated with u_{sa} and e_{sa} (see condition (b)), which we call the ***supplementary condition***, must not be forgotten. The patch test is associated to the satisfaction of such condition.

Convergence being achieved whenever $\mathrm{d}(e_s, e'_{sa})$ is of the order of $\sqrt{\ell}$, inequality (5.16) shows that convergence can be possible even if the $(n+1)$th and $(m+1)$th derivatives of the displacements or strains are not bounded. All it is needed for convergence, indeed, is the boundedness of $\ell^m E_{m+1}$ and $\ell^n U_{n+1}$. This indicates that convergence may still be possible even in cases in which the patch test is not passed.

Our aim being a limit analysis of the error, conditions (c) and (d) may be satisfied only in the limit, i.e. the requirement of the allowed fields being polynomial with arbitrary coefficients can be replaced by the one of the arbitrariness of the corresponding coefficients in the Taylor's expansions of such fields around some point within Ω^e. Similarly, in what concerns condition (c), the strain fields have not to be generated by the displacement fields: it suffices that the strains generated by the allowed displacements tend to be allowed as ℓ becomes smaller and smaller.

Allowed strain fields are used very often which equilibrate vanishing body forces. Only boundary values of the allowed displacements are then required for the determination of $\mathbf{D}^e$ with the help of eqn (3.20), so that the allowed displacement fields within Ω^e need not to be made explicit [7]. This can make more difficult, but of course not impossible, checking condition (c).

6. THE DISPLACEMENT AND STRAIN DERIVATIVE BOUNDEDNESS CRITERION

Our aim now is making the supplementary condition comparatively easy to check.

Admitting that the fields u_{sa} and e_{sa} are such that the derivatives of their components are of the same order as those associated with u'_{sa} and e'_{sa} (and this certainly happens if they are constructed like in Section 7), the supplementary condition is satisfied whenever the derivatives of order $n+1$ and $m+1$ of the displacements and strains associated with the approximate solution, are bounded. In other words, eqn (5.16) remains true if condition (b) of Theorem 5.3 is replaced by the following one:

"(b') the moduli of the derivatives of order respectively $n+1$ and $m+1$ of the displacements and strains associated with the approximate solution are bounded within each subdomain Ω^e".

This remark can be and has been applied by the author [11] in certain cases, namely when; (a) the fact that certain derivatives always vanish, or (b) the boundedness of the strain energy density, or (c) the boundedness of the body force density, or (d) the fact that the nodes are all of the same kind, imply the boundedness of certain derivatives of the displacements and strains.

A typical case is Morley's plate element [13] characterized by a quadratic transverse displacement field. As the element is non-hybrid and Kirchhoff's assumption is admitted, the strains (curvatures) become second derivatives of the transverse displacement. They are constant therefore within each element, and conditions (c) and (d) of Theorem 5.3 are respectively satisfied with $m = 0$ and $n = 2$. The supplementary condition (b') is certainly satisfied with $m = 0$ and $n = 0$, because the first derivatives of the allowed strains vanish and the first derivatives of the displacements are the rotations. Convergence is thus achieved regardless of the type of mesh.

In what concerns the BCIZ (Bazeley, Cheung, Irons and Zienkiewicz) triangular plate element,[4], the boundedness of the third derivatives of the transverse displacement can be concluded if the mesh is such that the nodes are all of the same kind (see[11]).

Another typical situation is the rectangular ACM (Adini, Clough, Melosh) element[14], the generalized displacements of which are the transverse displacement and the rotations at the corners. The allowed transverse displacements are of the form

$$w = P_3(x_1, x_2) + a_{31}x_1^3x_2 + a_{13}x_1x_2^3 \tag{6.1}$$

if axes are taken parallel to the axes (P_3 denotes a complete polynomial of the third degree with arbitrary coefficients). As the nodes, which may be assumed equidistant for sake of simplicity, are all of the same kind, and the generalized displacements are the nodal values of w and its first derivatives, the assumption of a smooth variation of such generalized displacements from node to node along the coordinate lines containing the nodes may lead to the conclusion that the derivatives w_{111}, $(w_2)_{11}$, $(w_1)_{22}$, w_{222}, $(w_2)_{111}$, $(w_1)_{222}$ associated with the approximate solution are bounded along the edges of the rectangles. The boundedness of the third derivatives of w (and, for physical reasons, of the second and first derivatives) associated with the approximate solution may be ensured then at any point within Ω^e and, the satisfaction of the supplementary condition thus being concluded, convergence is certain.

The boundedness of the $(n + 1)$th derivatives is however not always easy to predict, moreover if $n > O$, and this explains why the patch test can be so useful.

7. THE FORCE BOUNDEDNESS CRITERION

A second criterion for the fulfillement of the supplementary condition is presented in this Section. The justification of the patch test will be based on such criterion.

Let $\mathscr{E}_\pi$ denote a subset of $\mathscr{E}$ containing fields equilibrated by bounded systems of external forces, i.e. such that, if $e \in \mathscr{E}_\pi$, the norm $\|\Pi e\|_{\mathscr{F}}$ is bounded. Let $\mathscr{E}'_\pi \in \mathscr{E}'$ denote the set of the $I_\mathscr{E}$-images of the elements of $\mathscr{E}_\pi$.

Theorem 7.1: "The norms of the systems of discrete forces which equilibrate the $A_\mathscr{E}$-images of the elements of $\mathscr{E}_\pi$ are bounded."

Indeed, making

$$f = \Pi e \tag{7.1}$$

$$f' = \Pi' A_\mathscr{E} e \tag{7.2}$$

eqn (3.34) holds, so that, as Γ has been admitted to be bounded and $\|f\|_{\mathscr{F}}$ is bounded, $\|f'\|_{\mathscr{F}'}$ is also bounded.

A similar statement cannot be given for the $I_\mathscr{E}$-images of the elements of $\mathscr{E}_\pi$. In other words, the fact that a strain field e belongs to $\mathscr{E}_\pi$ does not ensure the boundedness of the norm $\|\Pi' I_\mathscr{E} e\|_{\mathscr{F}'}$.

The following theorem can be stated which is the foundation of the force boundedness criterion.

Theorem 7.2: "If the norms of the systems of discrete forces which equilibrate the $I_\mathscr{E}$-images of the elements of a certain subset $\mathscr{E}_\pi \subset \mathscr{E}$ are bounded, and field e_s belongs to $\mathscr{E}_\pi$, then the satisfaction of condition (a) of Theorem 5.3 implies the satisfaction of condition (b) (supplementary condition) of the same theorem".

Let us start the demonstration by constructing fields u_{sa} and e_{sa}.

Let $\mathscr{U}_{\mathscr{C}_0}$ denote the isocompatible subset of $\mathscr{U}$ corresponding to vanishing prescribed displacements on β_2 and vanishing displacement discontinuities on γ, and let N denote the number of degrees of freedom of the discretized structure (N is thus also the number of elements of vector $\mathbf{F}_1$).

Consider N displacement fields u_α such that: (i) $u_\alpha \in \mathcal{U}_{\mathscr{C}_0}$ and $\Delta u_\alpha \in \mathscr{E}_\pi$; (ii) all the derivatives of the components of the fields u_α are bounded on Ω; (iii) the N fields

$$u'_\alpha = I_\mathcal{U}(u_\alpha) \tag{7.3}$$

are linearly independent.

Each value of α corresponds to a certain degree of freedom and, therefore, to a certain node and a certain co-ordinate direction i.

Fields u_α may be for instance the continuous displacement fields actually introduced in the body by vanishing prescribed displacements on β_2, vanishing displacement discontinuities on γ, vanishing prescribed tractions on β_1, and body forces with density

$$f_{\alpha j} = e^{(\mathbf{x}-\mathbf{x}_\nu)(\mathbf{x}-\mathbf{x}_\nu)}\delta_{ij} \tag{7.4}$$

on Ω, where $\mathbf{x}$ represents a coordinate vector and ν and i denote the node and co-ordinate direction corresponding to α.

Let us call e_α the strain field

$$e_\alpha = \Delta u_\alpha. \tag{7.5}$$

As u_α belongs to $\mathcal{U}_{\mathscr{C}_0}$, e_α clearly belongs to $\mathscr{E}_{\mathscr{C}_0,0}$.

Fields u'_α and

$$e'_\alpha = I_\mathscr{E} e_\alpha \tag{7.6}$$

belong on the other hand to the N-dimensional subspaces $\mathcal{U}'_{\mathscr{C}'_0} \subset \mathcal{U}'$ and $\mathscr{E}'_{\mathscr{C}'_0,0'} \subset \mathscr{E}'$, i.e. to the isocompatible subsets of $\mathcal{U}'$ and $\mathscr{E}'$ corresponding to $\mathcal{U}_{\mathscr{C}_0}$ and $\mathscr{E}_{\mathscr{C}_0,0}$.

Now, as the values of the displacements at the nodes located on β_2 and of the nodal displacement discontinuities at the nodes located on γ are the same for the exact solution and for the approximate solution, the difference $u'_{sa} - I_\mathcal{U} u_s$ certainly belongs to $\mathcal{U}'_{\mathscr{C}'_0}$. Similarly, for these reasons and also because (3.28) holds, the difference $e'_{sa} - I_\mathscr{E} e_s$ belongs to $\mathscr{E}_{\mathscr{C}'_0,0'}$.

As, on the other hand, the N linearly independent fields $u'_\alpha \in \mathcal{U}'_{\mathscr{C}'_0}$ form a basis for the N-dimensional space $\mathcal{U}'_{\mathscr{C}'_0}$, a set of coefficients γ_α can be determined such that

$$u'_{sa} - I_\mathcal{U} u_s = \sum_\alpha \gamma_\alpha u'_\alpha. \tag{7.7}$$

Applying operator Δ' and using (3.29) and (7.5), we obtain also

$$e'_{sa} - I_\mathscr{E} e_s = \sum_\alpha \gamma_\alpha e'_\alpha. \tag{7.8}$$

Considering the linearity of operators $I_\mathcal{U}$ and $I_\mathscr{E}$, eqns (7.7) and (7.8) can be transformed into

$$u'_{sa} = I_\mathcal{U}\left(\sum_\alpha \gamma_\alpha u_\alpha + u_s\right) \tag{7.9}$$

$$e'_{sa} = I_\mathscr{E}\left(\sum_\alpha \gamma_\alpha e_\alpha + e_s\right) \tag{7.10}$$

which, compared with (4.12) and (4.13), show that u_{sa} and e_{sa} can be expressed by

$$u_{sa} = \sum_\alpha (\gamma_\alpha u_\alpha + u_s) \tag{7.11}$$

and

$$e_{sa} = \sum_\alpha (\gamma_\alpha e_\alpha + e_s). \tag{7.12}$$

Equations (7.11) and (7.12) make it clear that, if the coefficients γ_α are bounded, the derivatives associated with $u_{s\alpha}$ and $e_{s\alpha}$ satisfy the same boundedness conditions as those associated with u_s and e_s. The boundedness of the coefficients γ_α implies therefore the satisfaction of the supplementary condition, once condition (a) of Theorem 5.3 is also satisfied.

In order to demonstrate Theorem 7.2, it remains thus to prove that, if the norms of the systems of discrete forces which equilibrate the $I_{\mathscr{E}}$-images of fields e_α and e_s are bounded, then the coefficients γ_α are also bounded.

Applying indeed operator Π' to both sides of (7.10), we obtain, if the linearity of Π' is admitted,

$$\Pi' e'_{s\alpha} = \sum_\alpha \gamma_\alpha \Pi' I_{\mathscr{E}} e_\alpha + \Pi' I_{\mathscr{E}} e_s \tag{7.13}$$

and, therefore,

$$\left\| \sum_\alpha \gamma_\alpha \Pi' I_{\mathscr{E}} e_\alpha \right\|_{\mathscr{F}'} \leq \|\Pi' e'_{s\alpha}\|_{\mathscr{F}'} + \|\Pi' I_{\mathscr{E}} e_s\|_{\mathscr{F}'}. \tag{7.14}$$

As $\|\Pi' e'_{s\alpha}\|_{\mathscr{F}'}$ is bounded by virtue of Theorem 7.1 and $\|\Pi' I_{\mathscr{E}} e_s\|_{\mathscr{F}'}$ is assumed to be bounded, the left-hand side of (7.14) is bounded. Then, as the sets of discrete forces $\Pi' I_{\mathscr{E}} e_\alpha$ are linearly independent and their norms are also assumed to be bounded, the coefficients γ_α must be bounded and Theorem 7.2 is proved.

8. THE PATCH TEST

The application of the force boundedness criterion depends thus on predicting the boundedness of the norms of the systems of discrete forces which equilibrate, when the dimensions of the subdomains tend to zero, the $I_{\mathscr{E}}$-images of the elements of a subset $\mathscr{E}_\pi \subset \mathscr{E}$ to which fields e_α and e_s are supposed to belong.

The patch test is nothing else than a practical way for recognizing such boundedness for a subset $\mathscr{E}_\pi$ containing bounded strain fields with bounded first order derivatives.

This conclusion results from four theorems which will be established first for three-dimensional elasticity and later adapted to plates.

Let us consider a three-dimensional patch with diameter ℓ. ℓ is also the order of magnitude of the dimensions of any element in the patch. Displacements are supposed to be prescribed at the boundary nodes and discrete forces at the internal ones. A system of discrete external forces acting at the internal nodes is denoted by f'^p and the space of such systems by $\mathscr{F}'^p$. The strain field which such prescribed displacements and forces introduce in the elements is denoted by $e'^p \in \mathscr{E}'^p$. Index p refers to the patch and is used for distinguishing the fields or spaces defined on the domain $\Omega^p \subset \Omega$ occupied by the patch from those defined on the whole domain Ω.

Theorem 8.1: "If the strains associated with e'^p are of the order of ℓ, then, the norm of f'^p is of the order of ℓ^3".

Let us consider indeed a magnification of the patch with diameter L, i.e. obtained from the original patch by dividing each distance by the scale factor

$$\mu = \frac{\ell}{L}. \tag{8.1}$$

The modulus of elasticity E of the magnified patch is assumed to have the same value as the modulus of elasticity of the original patch. Both, E and L, are assumed bounded, together with their inverses.

If the displacements and strains of the magnified patch are bounded, the discrete forces acting at the nodes of the patch must be bounded because E is also bounded.

Let us revert now to the initial size of the patch by multiplying by μ every magnitude with the dimensions of a length. The forces are multiplied simultaneously by μ^2 (order of ℓ^2) in order that the modulus of elasticity ($[F][L]^{-2}$) keeps its value.† The displacements ($[L]$) become then of the order of ℓ and the strains ($[L][L]^{-1}$) maintain their values.

Keeping the values of the distances and of the modulus of elasticity, the forces vary proportionally to the displacements and strains, so that, if the displacements and strains are now multiplied by μ, becoming respectively of the order of ℓ^2 and ℓ, the forces are also multiplied by μ and become indeed of the order of ℓ^3.

Theorem 8.2: "If $\|f'^p\|_{\mathscr{F}'^p}$ is of the order of ℓ^3 whenever e'^p-image of any field with uniform strains, then it will be of the same order if e'^p is the $I_{\mathscr{E}^p}$-image of any field e^p such that the first derivatives of the strains are bounded".

Indeed, if such derivatives are bounded, the strains associated to e^p can be expressed by

$$\mathbf{e} = \mathbf{P}_0 + O(\ell) \tag{8.2}$$

where $\mathbf{P}_0$ denotes a constant term. As $\|f'^p\|_{\mathscr{F}'^p}$ is assumed of the order of ℓ^3 for such constant term, and was proved (Theorem 8.1) to be of the same order for the terms of the order of ℓ, the theorem is proved.

The patch test consists in imposing to the nodes on the boundary of the patch discrete displacements equal to the values at such nodes of arbitrary linear continuous displacement fields, and in checking if the displacements at the internal nodes, assumed free from external discrete forces, are also equal to the corresponding nodal values of the same fields.

In other words, the patch test consists in checking if the norm of the discrete force vector which equilibrates the $I_{\mathscr{E}}^p$-image of any uniform strain field on Ω^p is equal to zero.

Theorem 8.2 can take thus the following form.

Theorem 8.3: "If the patch passes the patch test, $\|f'^p\|_{\mathscr{F}'^p}$ is of the order of ℓ^3 whenever e'^p is the $I_{\mathscr{E}^p}$-image of a field e^p such that the first derivatives of the strains are bounded".

The next Theorem concerns the body as a whole.

Theorem 8.4: "If any patch in the body passes the patch test, then, the norm of the discrete force system f' which equilibrates the $I_{\mathscr{E}}$-image of any strain field with bounded first derivatives is bounded".

This theorem can easily be demonstrated if the body is decomposed into patches with diameter ℓ. As $\|f'\|_{\mathscr{F}'}$ is of the order of ℓ^3 for each of such patches and

$$\|f'\|_{\mathscr{F}'} = \sqrt{\sum_p \|f'^p\|^2_{\mathscr{F}'^p}}, \tag{8.3}$$

there follows that $\|f'\|_{\mathscr{F}'}$ is bounded.

Theorems 7.2 and 8.4 put together permit to conclude that, if any patch in the body passes the patch test, and the strains associated with the exact solution have bounded first order derivatives on Ω, the fulfillment of condition (a) of Theorem 5.3 implies the fulfillment of condition (b), i.e. of the supplementary condition. This means that passing the patch test has the same effect as keeping conformity, under the point of view of convergence and accuracy.

The adaptation of this reasoning to plane elasticity is simple. The adaption to plates offers some difficulty however because the deformation of a plate depends on two constants with different dimensions: the flexural rigidity $D = (Et^3/12(1-\nu^2))$ and the transverse shear rigidity Gt, with dimensions respectively $[D] = [F][L]$ and $[Gt] = [F][L]^{-1}$. It is clear that if the value of D is kept constant during the magnification, the value of Gt will vary and vice-versa.

Assuming however that the magnitudes with the dimensions of a force keep their value, the flexural rigidity and the tranverse shear rigidity become, in the magnified patch, equal to $D_0(\ell/L)$

†No precaution has to be taken for keeping the value of the Poisson's coefficient because this constant is dimensionless.

and $(Gt)_0(L/\ell)$, where D_0 and $(Gt)_0$ denote the flexural rigitity and the transverse shear rigidity of the initial patch. Assuming Kirchhoff's assumption to hold in the initial patch, $(Gt)_0$ being of the order of ℓ^{-1}, and supposing that D_0 does not vanish, the magnified patch presents an unbounded flexural rigidity and a bounded non-vanishing (Gt), so that bounded discrete displacements correspond still to bounded discrete forces and vice-versa.

As the forces keep their values, bounded discrete forces and nodal moments of the order of ℓ correspond to displacements of the order of ℓ and bounded rotations in the initial patch. Forces and moments respectively of the order of ℓ^2 and ℓ^3 correspond to displacements and rotations respectively of the order of ℓ^3 and ℓ^2, and to curvatures of the order of ℓ. Then, as the patch test consists now in applying to the nodes on the boundary of the patch discrete displacements corresponding to arbitrary quadratic displacement fields, Theorem 8.4 is still true.

9. CONCLUSIONS

The basic result of the present paper is inequality (5.16) which gives the expression for an upper bound of the discretization error associated with the use of hybrid structural elements.

Such expression may be used in an almost direct way if certain strain and displacement derivatives associated with the approximate solution can be predicted to remain bounded within the elements as the size of such elements becomes smaller and smaller. This has been done for different kinds of nonconforming elements whose ability for convergence could thus easily be established.

The difficulty in predicting the boundedness of such derivatives makes it necessary however to resort to special devices like the patch test.

It can be said in brief that convergence will be obtained if the completeness conditions are fulfilled and the patch test is passed. It was shown however in Section 5 that passing the patch test is not a necessary condition for convergence.

It was also proved in the paper that passing the (simple) patch test and satisfying completeness conditions of a certain order are sufficient condition for the error being of an order which depends on the order of the completeness conditions, so that the use of a higher-order patch test does not seem necessary for accuracy analyses.

Such conclusion cannot be surprising if it is remarked that, in the conforming case, the order of the error also depends exclusively on the order of the completeness condition, and not on the satisfaction of a higher-order compatibility condition associated with the continuity of some displacement derivatives across the element boundaries.

The author understands however that, although the order of the error is a limit concept to which the actual magnitude of the error is expected to be connected in the real cases in which the elements are indeed finite, it must not be confounded with such magnitude itself, so that the questions of the conditions in which and of the extent to which passing a higher-order patch test or satisfying a higher-order compatibility condition contribute for decreasing the error remain open and certainly deserve to be investigated.

Acknowledgement—The research reported in this paper was carried out at the Center of Mechanics and Materials of the Universities of Lisbon (CEMUL) and supported partly by the Institute Nacional de Investigacão Científica (INIC) and partly by the Calouste Gulbenkian Foundation.

REFERENCES

1. T. Miyoshy, Convergence of finite element solutions represented by a non-conforming basis. *Kumamoto J. Sci. Math.* **9**, 11–20 (1972).
2. J. A. Nitsche, Convergence of non-conforming methods. In *Mathematical Aspects of Finite Elements in Partial Differential Equations* (Edited by C. de Boor), pp. 15–53. Academic Press, New York (1974).
3. F. Kikushi, Approximation in finite element models—I. General theory for linear problems. *Report No.* 531 (1975), Institute of Space and Aeronautical Science, University of Tokyo, Tokyo (Sept. 1975).
4. G. R. Bazeley, Y. K. Cheung, B. M. Irons and O. C. Zienkiewicz, Triangular elements in plate bending. Conforming and non-conforming solutions. *Proc. of the 1st. Conf. on Matrix Methods in Structural Mechanics.* Wright-Patterson AFB (1965).
5. G. Strang and G. Fix, *An Analysis of the Finite Element Method.* Prentice-Hall, Englewood Cliffs (1973).
6. P. G. Ciarlet, Conforming and nonforming finite element methods for solving the plate problem. In *Conference on the Numerical Solution of Differential Equations* (Edited by G. A. Watson), pp. 21–31, Lecture Notes in Mathematics, Vol. 363. Springer-Verlag, Berlin (1974).
7. T. H. Pian, Derivation of element stiffness matrices by assumed stress distributions. *AIAA J.* 2(7) (1964)
8. E. R. Arantes Oliveira, Theoretical foundations of the finite element method. *Int. J. Solids Structures*, **4**, 929–952 (1968).

9. T. H. Pian and Pin-Tong, A variational principle and the convergence of a finite element method based on assumed stress distribution. *Int. J. Solids Structures* **5**, 463 (1969).
10. J. T. Oden and J. K. Lee, Theory of mixed and hybrid finite-element approximations in linear elasticity. *IUTAM/IMU Symp. Applications of Methods of Functional Analysis to Problems of Mechanics*, Marseilles, France (1975).
11. E. R. Arantes Oliveira, The convergence theorems and their role in the theory of structures. In the *Proc. of the IUTAM Symposium on High Speed Computing of Elastic Structures* (Edited by B. Fraeijs de Veubeke). Liège, Belgium (1970).
12. E. R. Arantes Oliveira, *A Theory of Variational Methods with Application to Finite Elements*. Lisbon (1972).
13. L. S. D. Morley, On the constant moment plate bending element. *J. Strain An.* **6**, 1 (1970).
14. A. Adini and R. W. Clough, Analysis of plate bending by the finite element method. *NSF Report* G, 7337 (1961).